Stability analysis of interconnected event-based control loops ⋆

Christian Stöcker * **Jan Lunze** * **Daniel Vey** *

* *Institute of Automation and Computer Control, Ruhr-Universität Bochum, Universitätsstr. 150, 44780 Bochum, Germany, (e-mail: {stoecker, lunze}@atp.rub.de, daniel.vey@rub.de)*

Abstract: In event-based control the feedback link within a control loop is only closed when an event indicates the need for information exchange among the sensors, controller and actuators to maintain a required loop performance. The event-based control loop is a hybrid dynamical system, which is characterized by a sequence of continuous state flows and discontinuous state jumps at the event times. This paper analyzes the stability of interconnected decentralized event-based control loops where events are triggered asynchronously. A stability criterion is derived using the comparison principle. It is shown that conditions that are sufficient to prove the stability of the continuous control system imply ultimate boundedness of the event-based state-feedback loop. The conservatism of the proposed stability test is evaluated for a thermofluid process.

1. INTRODUCTION

1.1 *Event-based control*

The event-based control paradigm aims at reducing the usage of the feedback link within a control loop to time instants at which an event indicates the need for an information exchange between sensors, controller and actuators in order to retain a desired closed-loop performance. The current output measurements are used to update the control input only at the event times, denoted as t_k, where $k \in \mathbb{N}_0$ is the event counter. The closed-loop system is a hybrid dynamical system the behavior of which is characterized by a continuous flow of the state between consecutive events and state resets at the event times, as explained by Donkers and Heemels [2010].

This paper analyzes the stability of N interconnected event-based state-feedback systems (Fig. 1). The overall system consists of

- N subsystems and their physical interconnections,
- N *event generators* (EG) that detect the trigger times t_k and
- N *control input generators* (CIG) each of which generates the control input in a decentralized manner using the received information at time t_k.

In Fig. 1 the solid lines represent continuous information transmissions, whereas the dashed lines indicate a communication that only occurs at the event times t_k.

The subsystem $i \in \mathcal{N} = \{1, \ldots, N\}$ is described by the linear state-space model

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}_i \boldsymbol{x}_i(t) + \boldsymbol{B}_i \boldsymbol{u}_i(t) + \boldsymbol{E}_i \boldsymbol{s}_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \tag{1a}$$

$$\boldsymbol{z}_i(t) = \boldsymbol{C}_{zi} \boldsymbol{x}_i(t), \tag{1b}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ denotes the plant state, $\boldsymbol{u}_i \in \mathbb{R}^{m_i}$ the control input, $\boldsymbol{s}_i \in \mathbb{R}^{p_i}$ the coupling input and $\boldsymbol{z}_i \in \mathbb{R}^{q_i}$ the coupling output. The subsystems are interconnected according to the relation

$$\boldsymbol{s}(t) = \boldsymbol{L}\boldsymbol{z}(t) \tag{2}$$

with

$$\boldsymbol{s}(t) = \left(\boldsymbol{s}_1^\top(t)\ \ldots \boldsymbol{s}_N^\top(t)\right)^\top \text{ and}$$
$$\boldsymbol{z}(t) = \left(\boldsymbol{z}_1^\top(t)\ \ldots \boldsymbol{z}_N^\top(t)\right)^\top. \tag{3}$$

As the event-based state-feedback controllers are designed under the assumption of vanishing interconnections ($\boldsymbol{s}_i = \boldsymbol{0}$), the isolated event-based control loops are stable. The objective of this paper is to analyze under what conditions on the interconnection $\boldsymbol{L}$, the stability of the subsystems with continuous decentralized control implies the stability of the event-based overall system.

1.2 *Literature review*

The analysis of interconnected event-based control loops has been addressed only in a few publications. In Wang and Lemmon [2008, 2010] and De Persis et al. [2011] the event-triggering conditions for the interconnected subsystems were defined such that a desired decay of a Lyapunov function for the overall system is guaranteed. These

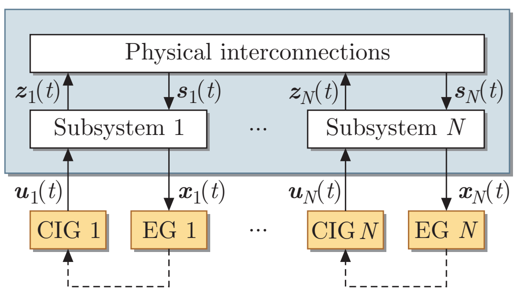

Fig. 1. Interconnected event-based control loops

⋆ This work was supported by the German Research Foundation (DFG) within the Priority Program 1305 "Control Theory of Digitally Networked Dynamical Systems".

approaches were shown to work for weakly coupled subsystems. Donkers and Heemels [2010, 2012] studied interconnected event-based control loops within the framework of impulsive systems, leading to stability conditions in the form of linear matrix inequalities. Mazo and Tabuada [2010] proved the interconnected event-based system to be input-to-state stabilizable under the requirement of synchronized measurements in all components. This work was extended by Mazo and Cao [2011] in order to allow for asynchronous updates, meaning that the event generation in one component of the event-based control loop only causes the update of sensor and/or actuator information in this and no other component.

All the above cited works have in common that a zero-order hold is used to keep the control input constant between two consecutive events. In contrast to that, this paper proposes a design approach that follows the idea of Lunze and Lehmann [2010] and yields control inputs with exponential flow. The presented analysis of the event-based overall system uses the comparison principle rather than Lyapunov functions for the stability test.

1.3 Outline of the paper

A state-feedback approach to decentralized event-based control is presented in Sec. 2. The stability of the composite system is analyzed in Sec. 3, where the comparison principle is used to derive a criterion to test the stability of the event-based overall system in dependence upon the physical interconnections. The main result of the paper is presented in Theorem 2 saying that the event-based overall system is ultimately bounded if the interconnections satisfy the same constraints that also guarantee asymptotic stability in the case of continuous state-feedback control. This stability condition is generalized in Sec. 4 which shows that the asymptotic stability of the continuous system implies ultimate boundedness of the event-based system. Section 5 illustrates the behavior of the decentralized event-based control scheme and the analysis results using the example of a thermofluid process.

1.4 Notation

For a scalar s, $|s|$ denotes the absolute value. For vectors and matrices the $|\cdot|$-operator applies to every element. $||\boldsymbol{x}||_\infty$ represents the uniform norm of $\boldsymbol{x}$. $\lambda_{\mathrm{p}}(\boldsymbol{A})$ denotes the Perron root of the matrix $\boldsymbol{A}$. The asterix $*$ symbolizes the convolution-operator, e. g.

$$\boldsymbol{G} * \boldsymbol{u} = \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{u}(\tau)\mathrm{d}\tau.$$

$\boldsymbol{A} = \mathrm{diag}\,(\boldsymbol{A}_i)$ denotes a block diagonal matrix having a main diagonal that consists of the matrices $\boldsymbol{A}_i$ with $i = 1, \ldots, N$.

2. DECENTRALIZED EVENT-BASED CONTROL

This paper analyzes the interconnected system shown in Fig. 1, where the control input generators and event generators for the subsystems are designed by using the method of Lunze and Lehmann [2010], which is applied here for the isolated subsystems

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}_i\boldsymbol{x}_i(t) + \boldsymbol{B}_i\boldsymbol{u}_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \tag{4}$$

for all $i \in \mathcal{N}$.

2.1 Control input generator

The i-th control input generator incorporates a model

$$\dot{\boldsymbol{x}}_{\mathrm{s}i}(t) = \bar{\boldsymbol{A}}_i\boldsymbol{x}_{\mathrm{s}i}(t), \quad \boldsymbol{x}_{\mathrm{s}i}(t_k^+) = \boldsymbol{x}_i(t_k) \tag{5}$$

of the decoupled continuous control loop, where $\boldsymbol{x}_{\mathrm{s}i} \in \mathbb{R}^{n_i}$ denotes the model state. It is assumed that a state feedback gain $\boldsymbol{K}_i$ exists such that the matrix

$$\bar{\boldsymbol{A}}_i = (\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_i)$$

is asymptotically stable. In (5) t_k^+ represents the right limit of the event time t_k at which the state of the model (5) is reset to the current plant state $\boldsymbol{x}_i(t_k)$. The model state $\boldsymbol{x}_{\mathrm{s}i}(t)$ is used to generate the control input $\boldsymbol{u}_i(t)$ to subsystem i according to

$$\boldsymbol{u}_i(t) = -\boldsymbol{K}_i\boldsymbol{x}_{\mathrm{s}i}(t). \tag{6}$$

2.2 Event generator

The event times t_k are determined in a decentralized manner by the event generators of the subsystems. Consider the i-th isolated subsystem (4). The application of the control input (6) yields the event-based closed-loop system

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) &= \boldsymbol{A}_i\boldsymbol{x}_i(t) - \boldsymbol{B}_i\boldsymbol{K}_i\boldsymbol{x}_{\mathrm{s}i}(t)\\ &= \bar{\boldsymbol{A}}_i\boldsymbol{x}_i(t) + \boldsymbol{B}_i\boldsymbol{K}_i\left(\boldsymbol{x}_i(t) - \boldsymbol{x}_{\mathrm{s}i}(t)\right).\end{aligned} \tag{7}$$

$\bar{\boldsymbol{A}}_i$ is stable and the behavior of (7) only depends upon the difference state

$$\boldsymbol{x}_{\Delta i}(t) = \boldsymbol{x}_i(t) - \boldsymbol{x}_{\mathrm{s}i}(t), \tag{8}$$

which should be bounded by means of the event generation. An event is triggered in the i-th subsystem whenever the relation

$$||\boldsymbol{x}_{\Delta i}(t)||_\infty = ||\boldsymbol{x}_i(t) - \boldsymbol{x}_{\mathrm{s}i}(t)||_\infty = \bar{e}_i \tag{9}$$

holds, where $\bar{e}_i \in \mathbb{R}_+$ denotes the event threshold for subsystem i. At this time t_k the current plant state $\boldsymbol{x}_i(t_k)$ is transmitted to the i-th control input generator and the state $\boldsymbol{x}_{\mathrm{s}i}$ of the model (5) used in both generators of subsystem i is reset.

Due to an initial reset of the model states $\boldsymbol{x}_{\mathrm{s}i}(0)$ at time $t = 0$, $\boldsymbol{x}_{\Delta i}(0) = \boldsymbol{0}$ holds for all $i \in \mathcal{N}$. In addition, each state reset causes $\boldsymbol{x}_{\Delta i}(t_k^+) = \boldsymbol{0}$, which implies the boundedness of the difference state (8) according to

$$|\boldsymbol{x}_{\Delta i}(t)| \leq (\bar{e}_i \ \ldots \ \bar{e}_i)^\top = \boldsymbol{e}_i, \quad \forall\, t \geq 0, \tag{10}$$

where $\boldsymbol{e}_i$ is a vector of dimension n_i.

2.3 Decentralized event-based control system

The event-based overall system consists of

- the plant (1), (2),
- N control input generators (5), (6) and
- N event generators each of which incorporates a model (5) of the corresponding subsystem and triggers an event whenever the condition (9) is satisfied.

In the overall system the state updates occur asynchronously since an event in subsystem i causes a reset in the control input generator and the event generator of this subsystem only.

Since the decentralized event-based controllers are only known to stabilize the isolated subsystems, the main analysis problem to be solved in the next section concerns the question how large the interconnections between the subsystems can be such that the stability of the overall system is guaranteed.

Remark 1. Under the assumption of bounded external inputs, the event-based state-feedback approach that is used for the design of the subsystems was shown in Lunze and Lehmann [2010] to yield a positive minimum inter-event time. Hence, given a limited number N of subsystems, the accumulation of infinite state resets in finite time, i. e. Zeno behavior (cf. Lunze and Lamnabhi-Lagarrigue [2009]) can be excluded for the event-based overall system on condition that it is stable.

3. STABILITY ANALYSIS

3.1 Comparison systems

The subsequently presented stability analysis makes use of comparison systems which yield upper bounds on the signals of the respective subsystems.

Definition 1. Consider the isolated subsystem (4) that has the state trajectory

$$\boldsymbol{x}_i(t) = \mathrm{e}^{\bar{\boldsymbol{A}}_i t}\boldsymbol{x}_{i0} + \boldsymbol{G}_{\mathrm{xu}i} * \boldsymbol{u}_i,$$

where

$$\boldsymbol{G}_{\mathrm{xu}i}(t) = \mathrm{e}^{\bar{\boldsymbol{A}}_i t}\boldsymbol{B}_i \tag{11}$$

is the impulse response matrix. The system

$$\boldsymbol{r}_i(t) = \bar{\boldsymbol{F}}_i(t)\,|\boldsymbol{x}_{i0}| + \bar{\boldsymbol{G}}_{\mathrm{xu}i} * |\boldsymbol{u}_i| \tag{12}$$

with $\boldsymbol{r}_i \in \mathbb{R}^{n_i}$ is called a *comparison system* of subsystem (4) if it satisfies the inequality

$$\boldsymbol{r}_i(t) \geq |\boldsymbol{x}_i(t)|\,, \quad \forall\, t \geq 0$$

for an arbitrary bounded input $\boldsymbol{u}_i(t)$.

A method for finding a comparison system is given in the following lemma.

Lemma 1. (Lunze [1992]). The system (12) is a comparison system of the system (4) if and only if the matrices $\boldsymbol{F}_i(t)$ and $\boldsymbol{G}_{\mathrm{xu}i}(t)$ satisfy the relations

$$\bar{\boldsymbol{F}}_i(t) \geq \left|\mathrm{e}^{\bar{\boldsymbol{A}}_i t}\right|, \quad \bar{\boldsymbol{G}}_{\mathrm{xu}i}(t) \geq |\boldsymbol{G}_{\mathrm{xu}i}(t)|\,, \quad \forall\, t \geq 0$$

where the impulse response matrix $\boldsymbol{G}_{\mathrm{xu}i}(t)$ is defined in Eq. (11).

3.2 Analysis of the continuous overall system

This section provides a stability analysis for the continuous overall system with continuous state feedback

$$\boldsymbol{u}_i(t) = -\boldsymbol{K}_i\boldsymbol{x}_i(t), \quad \forall\, i \in \mathcal{N}. \tag{13}$$

The controlled subsystems (1), (13) are described by the state-space models

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{x}}_i(t) &= \bar{\boldsymbol{A}}_i\hat{\boldsymbol{x}}_i(t) + \boldsymbol{E}_i\boldsymbol{s}_i(t), \quad \hat{\boldsymbol{x}}_i(0) = \boldsymbol{x}_{i0} \\ \boldsymbol{z}_i(t) &= \boldsymbol{C}_{\mathrm{z}i}\hat{\boldsymbol{x}}_i(t) \end{aligned} \tag{14}$$

for $i \in \mathcal{N}$, where $\hat{\boldsymbol{x}}_i \in \mathbb{R}^{n_i}$ denotes the state of the i-th continuous feedback loop. The behavior of subsystem i from input $\boldsymbol{s}_i$ to state $\hat{\boldsymbol{x}}_i$ is given by

$$\hat{\boldsymbol{x}}_i(t) = \mathrm{e}^{\bar{\boldsymbol{A}}_i t}\boldsymbol{x}_{i0} + \boldsymbol{G}_{\mathrm{xs}i} * \boldsymbol{s}_i$$

with

$$\boldsymbol{G}_{\mathrm{xs}i}(t) = \mathrm{e}^{\bar{\boldsymbol{A}}_i t}\boldsymbol{E}_i. \tag{15}$$

Hence,

$$\boldsymbol{r}_{\hat{\mathrm{x}}i}(t) = \bar{\boldsymbol{F}}_i(t)\,|\boldsymbol{x}_{i0}| + \bar{\boldsymbol{G}}_{\mathrm{xs}i} * |\boldsymbol{s}_i| \tag{16}$$

with

$$\bar{\boldsymbol{F}}_i(t) = \left|\mathrm{e}^{\bar{\boldsymbol{A}}_i t}\right|, \quad \bar{\boldsymbol{G}}_{\mathrm{xs}i}(t) = |\boldsymbol{G}_{\mathrm{xs}i}(t)| \tag{17}$$

is a comparison system for the original system (14).

In order to investigate asymptotic stability of the overall system, let

$$\boldsymbol{r}_{\hat{\mathrm{x}}}(t) = \left(\boldsymbol{r}_{\hat{\mathrm{x}}1}^\top(t)\ \dots\ \boldsymbol{r}_{\hat{\mathrm{x}}N}^\top(t)\right)^\top, \quad \boldsymbol{x}_0 = \left(\boldsymbol{x}_{10}^\top\ \dots\ \boldsymbol{x}_{N0}^\top\right)^\top$$

and

$$\bar{\boldsymbol{F}}(t) = \mathrm{diag}\left(\bar{\boldsymbol{F}}_i(t)\right), \quad \bar{\boldsymbol{G}}_{\mathrm{xs}}(t) = \mathrm{diag}\left(\bar{\boldsymbol{G}}_{\mathrm{xs}i}(t)\right),$$

which yields

$$\boldsymbol{r}_{\hat{\mathrm{x}}}(t) = \bar{\boldsymbol{F}}(t)\,|\boldsymbol{x}_0| + \bar{\boldsymbol{G}}_{\mathrm{xs}} * |\boldsymbol{s}| \tag{18}$$

with $\boldsymbol{s}(t)$ defined in (3). With the upper bound

$$|\boldsymbol{s}(t)| \leq |\boldsymbol{L}|\,|\boldsymbol{z}(t)|$$

on the interconnection relation (2), where $\boldsymbol{z}(t)$ is defined in (3) and

$$|\boldsymbol{z}(t)| \leq |\boldsymbol{C}_{\mathrm{z}}|\,|\hat{\boldsymbol{x}}(t)|$$

where $\boldsymbol{C}_{\mathrm{z}} = \mathrm{diag}\,(\boldsymbol{C}_{\mathrm{z}i})$, the relation

$$\boldsymbol{r}_{\hat{\mathrm{x}}}(t) = \bar{\boldsymbol{F}}\,|\boldsymbol{x}_0| + \bar{\boldsymbol{G}}_{\mathrm{xs}}\,|\boldsymbol{L}|\,|\boldsymbol{C}_{\mathrm{z}}| * |\hat{\boldsymbol{x}}| \geq |\hat{\boldsymbol{x}}(t)|$$

follows from (18). The last inequality is an implicit bound on the overall state $\hat{\boldsymbol{x}}(t)$. An explicit statement in terms of the initial condition is obtained by means of the comparison principle, Lunze [1992]. Accordingly,

$$\boldsymbol{r}_{\hat{\mathrm{x}}}(t) = \boldsymbol{G} * \bar{\boldsymbol{F}}\,|\boldsymbol{x}_0| \geq |\hat{\boldsymbol{x}}(t)| \tag{19}$$

holds, where the impulse response matrix $\boldsymbol{G}(t)$ satisfies the relation

$$\boldsymbol{G}(t) = \delta(t)\boldsymbol{I} + \bar{\boldsymbol{G}}_{\mathrm{xs}}\,|\boldsymbol{L}|\,|\boldsymbol{C}_{\mathrm{z}}| * \boldsymbol{G} \tag{20}$$

with $\delta(t)$ representing the Dirac impulse. Note that according to (19) the continuous system (1), (2), (13) is asymptotically stable if $\boldsymbol{r}_{\hat{\mathrm{x}}}(t)$ converges to zero. To use this fact as a stability criterion, it has to be shown that for the impulse response matrix (20) of the comparison system (19) the condition

$$\int_0^\infty \boldsymbol{G}(t)\mathrm{d}t < \infty \tag{21}$$

holds. Equation (20) yields

$$\begin{aligned} \int_0^\infty \boldsymbol{G}(t)\ \mathrm{d}t &= \int_0^\infty \delta(t)\boldsymbol{I}\mathrm{d}t + \int_0^\infty \bar{\boldsymbol{G}}_{\mathrm{xs}}\,|\boldsymbol{L}|\,|\boldsymbol{C}_{\mathrm{z}}| * \boldsymbol{G}\ \mathrm{d}t \\ &= \boldsymbol{I} + \int_0^\infty \bar{\boldsymbol{G}}_{\mathrm{xs}}(t)\,|\boldsymbol{L}|\,|\boldsymbol{C}_{\mathrm{z}}|\ \mathrm{d}t\ \int_0^\infty \boldsymbol{G}(t)\ \mathrm{d}t, \end{aligned}$$

from which the relation

$$\int_0^\infty \boldsymbol{G}(t)\ \mathrm{d}t\left(\boldsymbol{I} - \int_0^\infty \bar{\boldsymbol{G}}_{\mathrm{xs}}(t)\,|\boldsymbol{L}|\,|\boldsymbol{C}_{\mathrm{z}}|\ \mathrm{d}t\right) = \boldsymbol{I} \tag{22}$$

follows. Hence, a matrix $\boldsymbol{G}(t)$ satisfying (21) exists if the condition

$$\lambda_{\mathrm{p}}\left(\int_0^\infty \bar{\boldsymbol{G}}_{\mathrm{xs}}(t)\,|\boldsymbol{L}|\,|\boldsymbol{C}_{\mathrm{z}}|\ \mathrm{d}t\right) < 1 \tag{23}$$

is fulfilled. The following theorem summarizes the resulting stability test.

Theorem 1. Consider the decentralized continuous control loops (1), (13) that are connected over the interconnection relation (2). If the inequality (23) is satisfied, the overall system is asymptotically stable.

Remark 2. Note that the stability criterion (23) is a sufficient condition that can be regarded as a small-gain theorem. Since norm bounds on the impulse response matrix of the overall system, as well as on the coupling matrix $\boldsymbol{L}$ are used, the stability test might fail even if the overall system is in fact stable.

3.3 Analysis of the event-based overall system

This section extends the previously developed stability analysis to the event-based control system. Due to the event-based sampling the state of the overall hybrid system cannot be expected to converge asymptotically to the origin, but to a bounded set $\mathcal{X} \subset \mathbb{R}^n$. Hence, the subsequent analysis investigates the stability of the event-based control system in the sense of ultimate boundedness, Khalil [2002].

Definition 2. The event-based control system (1), (2), (5), (6), (9) is said to be globally ultimately bounded (GUB) if there exists a bounded set $\mathcal{X}$ and a time $\bar{t}$ such that for all initial states $\boldsymbol{x}_0$ the following relation holds:

$$\exists \bar{t} : \boldsymbol{x}(t) \in \mathcal{X}, \quad \forall t \geq \bar{t}.$$

The i-th subsystem (1a) with event-based control (5), (6) is described by the state-space model

$$\begin{aligned} \dot{\boldsymbol{x}}_i(t) &= \bar{\boldsymbol{A}}_i \boldsymbol{x}_i(t) + \boldsymbol{B}_i \boldsymbol{K}_i \boldsymbol{x}_{\Delta i}(t) + \boldsymbol{E}_i \boldsymbol{s}_i(t), \\ \boldsymbol{x}_i(0) &= \boldsymbol{x}_{i0}, \\ \boldsymbol{z}_i(t) &= \boldsymbol{C}_{\mathrm{z}i} \boldsymbol{x}_i(t) \end{aligned}$$

which yields the state trajectory

$$\boldsymbol{x}_i(t) = \mathrm{e}^{\bar{\boldsymbol{A}}_i t} \boldsymbol{x}_{i0} + \boldsymbol{G}_{\mathrm{xx}i} * \boldsymbol{x}_{\Delta i} + \boldsymbol{G}_{\mathrm{xs}i} * \boldsymbol{s}_i$$

with

$$\boldsymbol{G}_{\mathrm{xx}i}(t) = \mathrm{e}^{\bar{\boldsymbol{A}}_i t} \boldsymbol{B}_i \boldsymbol{K}_i$$

and the state difference $\boldsymbol{x}_{\Delta i}(t)$ and the impulse response matrix $\boldsymbol{G}_{\mathrm{xs}i}(t)$ defined in (8) or (15), respectively. An upper bound on the state $\boldsymbol{x}_i(t)$ is obtained by means of the comparison system

$$\boldsymbol{r}_{\mathrm{x}i}(t) = \bar{\boldsymbol{F}}_i(t) \left|\boldsymbol{x}_{i0}\right| + \bar{\boldsymbol{G}}_{\mathrm{xx}i} * \left|\boldsymbol{x}_{\Delta i}\right| + \bar{\boldsymbol{G}}_{\mathrm{xs}i} * \left|\boldsymbol{s}_i\right| \tag{24}$$

with

$$\bar{\boldsymbol{G}}_{\mathrm{xx}i}(t) = \left|\boldsymbol{G}_{\mathrm{xx}i}(t)\right|$$

and $\bar{\boldsymbol{F}}_i(t)$ and $\bar{\boldsymbol{G}}_{\mathrm{xs}i}(t)$ given in (17). As the difference state $\boldsymbol{x}_{\Delta i}(t)$ is bounded according to Eq. (10) the following relation holds:

$$\bar{\boldsymbol{G}}_{\mathrm{xx}i} * \left|\boldsymbol{x}_{\Delta i}\right| = \int_0^t \bar{\boldsymbol{G}}_{\mathrm{xx}i}(t-\tau) \left|\boldsymbol{x}_{\Delta i}(\tau)\right| \mathrm{d}\tau \leq \boldsymbol{e}_{\max i}$$

with

$$\boldsymbol{e}_{\max i} = \int_0^\infty \bar{\boldsymbol{G}}_{\mathrm{xx}i}(\tau) \mathrm{d}\tau \cdot \boldsymbol{e}_i. \tag{25}$$

Given (25) the comparison system (24) can be reformulated as

$$\boldsymbol{r}_{\mathrm{x}i}(t) = \bar{\boldsymbol{F}}_i(t) \left|\boldsymbol{x}_{i0}\right| + \boldsymbol{e}_{\max i} + \bar{\boldsymbol{G}}_{\mathrm{xs}i} * \left|\boldsymbol{s}_i\right|.$$

Note that the difference between the upper bound on the behavior of the continuous system (16) and the event-based system (24) is only the additional term $\boldsymbol{e}_{\max i}$ defined in (25), whereas the influence of $\boldsymbol{s}_i(t)$ on the state $\boldsymbol{x}_i(t)$ remains unchanged.

A comparison system for the overall event-based control loop is obtained in the same way as for the continuous system:

$$\boldsymbol{r}_{\mathrm{x}}(t) = \bar{\boldsymbol{F}}(t) \left|\boldsymbol{x}_0\right| + \boldsymbol{e}_{\max} + \bar{\boldsymbol{G}}_{\mathrm{xs}} \left|\boldsymbol{L}\right| \left|\boldsymbol{C}_{\mathrm{z}}\right| * \left|\boldsymbol{x}\right| \geq \left|\boldsymbol{x}(t)\right| \tag{26}$$

with

$$\boldsymbol{e}_{\max} = \left(\boldsymbol{e}_{\max 1}^\top \ \dots \ \boldsymbol{e}_{\max N}^\top\right)^\top.$$

The comparison principle is applied to Eq. (26) in order to obtain an explicit formulation of the upper bound on the state $\boldsymbol{x}(t)$:

$$\boldsymbol{r}_{\mathrm{x}}(t) = \boldsymbol{G} * \left(\bar{\boldsymbol{F}} \left|\boldsymbol{x}_0\right| + \boldsymbol{e}_{\max}\right) \geq \left|\boldsymbol{x}(t)\right|. \tag{27}$$

Here, the impulse response matrix $\boldsymbol{G}(t)$ of the comparison system is given by Eq. (20) and is, hence, the same matrix as for the continuous control loop. Note that the term $\bar{\boldsymbol{F}} \left|\boldsymbol{x}_0\right|$ asymptotically vanishes, whereas $\boldsymbol{e}_{\max}$ is non-vanishing. Therefore, the event-based overall system is ultimately bounded if the impulse response matrix $\boldsymbol{G}(t)$ satisfies the condition (21) which is true if the inequality (23) holds. These results are summarized in the following theorem.

Theorem 2. The decentralized event-based control loop (1), (2), (5), (6), (9) is ultimately bounded if the stability condition (23) is satisfied.

In summary, Eq. (23) represents a sufficient condition for the asymptotic stability of the continuous closed-loop system and for the ultimate boundedness of the event-based closed-loop system.

3.4 Ultimate bound

This section derives explicit bounds on the set $\mathcal{X}$ in which the state $\boldsymbol{x}(t)$ of the event-based overall system remains for large time t. It is assumed that the sufficient stability condition (23) is satisfied. According to the comparison system (27), the state $\boldsymbol{x}(t)$ of the event-based overall system is bounded by

$$\left|\boldsymbol{x}(t)\right| \leq \boldsymbol{r}_{\mathrm{x}}(t) = \boldsymbol{G} * \bar{\boldsymbol{F}} \left|\boldsymbol{x}_0\right| + \boldsymbol{G} * \boldsymbol{e}_{\max}.$$

The first term depends upon the initial state $\boldsymbol{x}_0$ and asymptotically converges to the origin, while the second term does not vanish. Hence, a limit for $\boldsymbol{r}_{\mathrm{x}}(t)$ is given by

$$\boldsymbol{r}_{\mathrm{x}}(t) \xrightarrow{t\to\infty} \int_0^\infty \boldsymbol{G}(t) \mathrm{d}t \, \boldsymbol{e}_{\max} = \boldsymbol{b},$$

where the vector $\boldsymbol{b}$ is referred to as *ultimate bound*. Note that a value for the integral over the impulse response matrix $\boldsymbol{G}(t)$ follows from Eq. (22) which yields

$$\int_0^\infty \boldsymbol{G}(t) \mathrm{d}t = \left(\boldsymbol{I} - \int_0^\infty \bar{\boldsymbol{G}}_{\mathrm{xs}}(t) \left|\boldsymbol{L}\right| \left|\boldsymbol{C}_{\mathrm{z}}\right| \mathrm{d}t\right)^{-1} = \boldsymbol{\Gamma}.$$

Thus, the ultimate bound $\boldsymbol{b}$ is

$$\boldsymbol{b} = \boldsymbol{\Gamma} \boldsymbol{e}_{\max} \tag{28}$$

and

$$\boldsymbol{x}(t) \xrightarrow{t\to\infty} \mathcal{X} = \left\{\boldsymbol{x} \middle| \left|\boldsymbol{x}\right| \leq \boldsymbol{b}\right\} \tag{29}$$

holds.

Theorem 3. Consider the event-based overall system (1), (2), (5), (6), (9) that satisfies the stability condition (23). Then (28), (29) define the set $\mathcal{X}$ with ultimate bound $\boldsymbol{b}$.

4. GENERALIZED STABILITY CONDITION

This section derives an alternative stability condition for the event-based control loop. It assumes that the continuous decentralized controller is known to stabilize the plant (1), (2) whether or not it satisfies the condition (23). It shows that the event-based implementation of the controller, following the design described in Sec. 2, leads to a system that is ultimately bounded.

The overall plant is described by the state-space model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

with $\boldsymbol{x}(t) = \big(\boldsymbol{x}_1^\top(t) \ \dots \boldsymbol{x}_N^\top(t)\big)^\top$,

$$\boldsymbol{A}(i,j) = \begin{cases} \boldsymbol{A}_i & \text{for } i = j \\ \boldsymbol{E}_i \boldsymbol{L}_{ij} \boldsymbol{C}_{zj} & \text{for } i \neq j \end{cases}$$

and $\boldsymbol{B} = \operatorname{diag}(\boldsymbol{B}_i)$. Assume that the state-feedback gain

$$\boldsymbol{K} = \operatorname{diag}(\boldsymbol{K}_i), \quad i = 1, \dots, N, \tag{30}$$

is designed such that the matrix $\bar{\boldsymbol{A}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$ is stable. In the event-based control system the input is generated by means of the model

$$\dot{\boldsymbol{x}}_{\mathrm{s}}(t) = \bar{\boldsymbol{A}}\boldsymbol{x}_{\mathrm{s}}(t), \quad \boldsymbol{x}_{\mathrm{s}i}(t_k^+) = \boldsymbol{x}_i(t_k)$$
$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}_{\mathrm{s}}(t)$$

which yields the closed-loop system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}_{\mathrm{s}}(t) \\ &= \bar{\boldsymbol{A}}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{K}\big(\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\big). \end{aligned} \tag{31}$$

As the deviation between the plant state $\boldsymbol{x}_i(t)$ and the model state $\boldsymbol{x}_{\mathrm{s}i}(t)$ is bounded by means of the event generation (9) for all $i \in \mathcal{N}$,

$$||\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)||_\infty \leq \max_{i \in \mathcal{N}} \bar{e}_i$$

follows, which implies ultimate boundedness of the event-based control loop (31).

Theorem 4. Given a decentralized state-feedback gain (30) that yields a continuous closed-loop system which is asymptotically stable. Then the event-based implementation of this controller in form of the decentralized event-based control stations (5), (6), that are reset each time the condition (9) holds, results in an ultimately bounded system.

Theorem 4 states that the event-based implementation of a decentralized state-feedback controller as proposed in Sec. 2 does not impose more restrictions on the interconnections between the subsystems than are claimed for asymptotic stability of the continuous overall system. That means that any condition that is sufficient to guarantee asymptotic stability of the continuous system is concomitantly sufficient to ensure ultimate boundedness of the event-based system.

5. EXAMPLE

5.1 Thermofluid process

The proposed decentralized event-based control approach is now applied to a thermofluid process depicted in Fig. 2. The process consists of two batch reactors T_1 and T_2 each of which is fed by the water supply S via the valves V_1 and V_2. The inflow can be controlled by means of the opening angles u_{V1} and u_{V2} of the respective valve. The outflow from T_1 and T_2 to W_1 or W_2, respectively, is constant. The temperature in each reactor can be increased by means of heating rods using the control signals u_{H1} and u_{H2}. Both reactors are defined as subsystems which are coupled

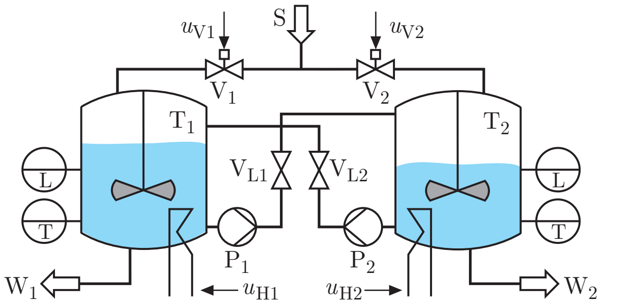

Fig. 2. Thermofluid process

through flows from T_1 to T_2 and vice versa. The strength of these couplings depend upon the opening angels of the valves V_{L1} and V_{L2}. The control aim is to keep the temperature ϑ and level l of the liquid constant in both reactors, using decentralized event-based controllers.

With the states

$$\boldsymbol{x}_1 = (l_{\mathrm{T1}} \ \vartheta_{\mathrm{T1}})^\top, \quad \boldsymbol{x}_2 = (l_{\mathrm{T2}} \ \vartheta_{\mathrm{T2}})^\top$$

and control inputs

$$\boldsymbol{u}_1 = (u_{\mathrm{V1}} \ u_{\mathrm{H1}})^\top, \quad \boldsymbol{u}_2 = (u_{\mathrm{V2}} \ u_{\mathrm{H2}})^\top$$

the system is described by the linear model (1) with

$$\begin{aligned} \boldsymbol{A}_1 &= 10^{-3}\begin{pmatrix} -1.45 & 0 \\ -45.6 & -5.34 \end{pmatrix}, & \boldsymbol{A}_2 &= 10^{-3}\begin{pmatrix} -0.84 & 0 \\ -38.2 & -6.62 \end{pmatrix} \\ \boldsymbol{B}_1 &= 10^{-3}\begin{pmatrix} 3.61 & 0 \\ -184 & 28.7 \end{pmatrix}, & \boldsymbol{B}_2 &= 10^{-3}\begin{pmatrix} 2.01 & 0 \\ -103 & 31.6 \end{pmatrix} \\ \boldsymbol{E}_1 &= 10^{-3}\,(0 \ 17.5)^\top, & \boldsymbol{E}_2 &= 10^{-3}\,(0 \ 15.2)^\top \\ \boldsymbol{C}_{\mathrm{z1}} &= (0 \ 1), & \boldsymbol{C}_{\mathrm{z2}} &= (0 \ 1) \end{aligned}$$

which is valid around the setpoint

$$\bar{\boldsymbol{x}}_1 = (40 \ 318)^\top, \qquad \bar{\boldsymbol{x}}_2 = (30 \ 308)^\top$$

where the level and the temperature are measured in cm and K, respectively, and

$$\bar{\boldsymbol{u}}_1 = (0.19 \ 1.83)^\top, \qquad \bar{\boldsymbol{u}}_2 = (0.48 \ 1.10)^\top$$

where the first value corresponds to the opening angle of the respective valve and the second value represents the number of active heating rods. The interconnection between both subsystems is modeled by the relation (2) with

$$\boldsymbol{L} = \begin{pmatrix} 0 & l_{12} \\ l_{21} & 0 \end{pmatrix}, \quad l_{12}, l_{21} \in [0, 1],$$

where l_{12} and l_{21} represent the opening angels of the valves V_{L2} or V_{L1}, respectively. The decentralized state-feedback controller $\boldsymbol{K} = \operatorname{diag}(\boldsymbol{K}_1, \boldsymbol{K}_2)$ with

$$\boldsymbol{K}_1 = \begin{pmatrix} 4.03 & 0 \\ 25.9 & 0.12 \end{pmatrix}, \quad \boldsymbol{K}_2 = \begin{pmatrix} 9.44 & 0 \\ 30.8 & 0.32 \end{pmatrix}$$

ensures stability of the decoupled subsystems, i. e. $l_{12} = l_{21} = 0$. For the event-based implementation of this controller the event thresholds

$$\bar{e}_1 = \bar{e}_2 = 0.5$$

are chosen.

5.2 Stability analysis of the thermofluid process

The following analysis investigates the stability of the event-based overall system under the assumption of symmetric couplings $l_{12} = l_{21} = l_{\mathrm{sym}}$. The evaluation of the stability test (23) for all possible symmetric couplings $l_{\text{sym}} \in [0, 1]$ shows that the test fails for interconnections that are larger than the critical value $l_{\text{crit}} = 0.741$. The analysis of the eigenvalues of the continuous control loop reveals that the overall system in fact becomes unstable for $l_{\text{sym}} \geq 0.743$ which, according to Theorem 4, implies instability of the event-based control loop. Hence, the stability test (23) yields excellent results if the impulse response matrices are largely positive such that the $|\cdot|$-operator has minor effect on the estimation of the systems behavior.

Given the coupling $l_{\text{sym}} = 0.3$ for which the stability test proved the system to be ultimately bounded, the set $\mathcal{X}$ is given by Eq. (29) with ultimate bound

$$\boldsymbol{b} = (0.46\ 4.04\ 0.48\ 2.50)^{\top} \tag{32}$$

derived according to (28).

5.3 Simulation results

The behavior of the event-based overall system is now investigated for the coupling $l_{\text{sym}} = 0.3$ and initial condition

$$\boldsymbol{x}_1(0) = (-5\ -1)^{\top}, \quad \boldsymbol{x}_2(0) = (5\ 1)^{\top}.$$

Figure 3 depicts the simulation results where the left-hand side shows the trajectories of the level l_{T1} and temperature ϑ_{T1} (first and second figure from top) of the liquid in reactor T_1 and the right-hand side gives the respective trajectories of the level l_{T2} and temperature ϑ_{T2} in reactor T_2. The solid and the dashed lines represents the plant states or the model states, respectively. The model state in each subsystem is reset whenever an event is triggered. The generated events are symbolized by the stems that are illustrated in the figures at the bottom. In this example event generation is solely caused by a difference between the temperatures of the plant and the model. This is due to the fact that the symmetric interconnection has no influence on the behavior of the level in each reactor. The feedback communication at the indicated event times is sufficient to control the overall system close to the setpoint.

Figure 4 shows the state trajectories for reactor T_1 (left-hand side) and for reactor T_2 (right-hand side). The set $\mathcal{X}$ that follows from the ultimate bound (32) is highlighted in grey in both plots. The overall system is clearly ultimately bounded for $l_{\text{sym}} = 0.3$, however, the derived ultimate bound $\boldsymbol{b}$ is conservative, as the state trajectories converge close to the setpoint.

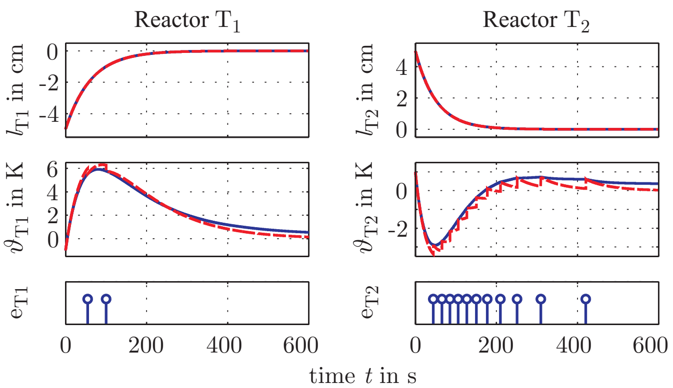

Fig. 3. Behavior of the event-based overall system

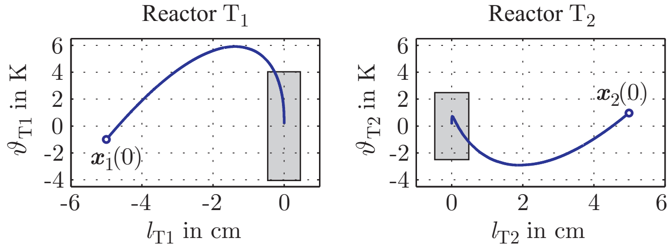

Fig. 4. Ultimate boundedness of the state trajectories

6. CONCLUSION

The paper has presented two stability tests for decentralized event-based control systems both showing that continuous decentralized controllers can be emulated by event-based controllers with arbitrary accuracy if the event generators and the control input generators are designed for the isolated subsystems by means of the method described by Lunze and Lehmann [2010]. If the continuous system is asymptotically stable, the event-based system is ultimately bounded, where the bound depends upon the event thresholds.

REFERENCES

C. De Persis, R. Sailer, and F. Wirth. On a small-gain approach to distributed event-triggered control. In *18th IFAC World Congress*, pages 2401–2406, Milan, Italy, 2011.

M.C.F. Donkers and W.P.M.H. Heemels. Output-Based Event-Triggered Control with Guaranteed $\mathcal{L}_\infty$-gain and Improved Event-Triggering. In *49th IEEE Conference on Decision and Control*, pages 3246–3251, Atlanta, Georgia, USA, 2010.

M.C.F. Donkers and W.P.M.H. Heemels. Output-Based Event-Triggered Control with Guaranteed $\mathcal{L}_\infty$-gain and Improved and Decentralised Event-Triggering. To appear, 2012.

H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, 3rd. edition, 2002.

J. Lunze. *Feedback Control of Large-Scale Systems*. Prentice Hall, Upper Saddle River, NJ, 1992.

J. Lunze and F. Lamnabhi-Lagarrigue, editors. *Handbook of Hybrid Systems Control*. Cambridge University Press, 2009.

J. Lunze and D. Lehmann. A state-feedback approach to event-based control. *Automatica*, 46(1):211–215, 2010.

M. Mazo and M. Cao. Decentralized Event-triggered Control with Asynchronous Updates. In *50th IEEE Conference on Decision and Control and European Control Conference 2011*, Orlando, Floriada, USA, 2011.

M. Mazo and P. Tabuada. Decentralized event-triggered control over wireless sensor/actuator networks. arXiv:1004.0477, 2010.

X. Wang and M.D. Lemmon. Decentralized Event-Triggered Broadcasts over Networked Control Systems. In *Hybrid Systems: Computation and Control*, pages 674–677, St. Louis, Missouri, USA, 2008.

X. Wang and M.D. Lemmon. Event-Triggering in Distributed Networked Control Systems. *IEEE Transactions on Automatic Control*, 56(3):586–601, 2010.