Event-based control with incomplete state measurement and guaranteed performance

Christian Stöcker * **Jan Lunze** *

* *Ruhr-Universität Bochum, 44780 Bochum, Germany,*
(e-mail: {stoecker, lunze}@atp.rub.de)

Abstract: The present paper proposes a novel method for the design of event-based controllers for the disturbance rejection in systems for which only a part of the state is measurable. The design approach is based on the decomposition of the system into subsystems where the state is either measurable or not accessible for measurement. The triggering conditions of the event-based controllers solely depend upon the measurable states. The idea of the design approach is to adjust the triggering conditions such that the deviation between the disturbance rejection behavior of the event-based control system and a continuous-time state-feedback system is less than a desired bound, which is defined as level of performance. This design method is formulated as a linear programming problem. The design of the triggering conditions and the behavior of the event-based control system is illustrated for an interconnected two-tank system.

1. INTRODUCTION

1.1 Event-based control

This paper investigates event-based disturbance rejection for linear systems, where the control performance is measured by the maximum deviation between the disturbance rejection behavior of the event-based control loop to be designed and a continuous reference system. In the existing literature the boundedness of this difference and, therefore, a desired performance is obtained for plants with measurable state $\boldsymbol{x}(t)$. This paper removes this assumption and considers plants for which only a part of the state is measurable.

In more detail, interconnected systems are studied that can be decomposed into subsystems $\Sigma_i, i \in \mathcal{N} = \{1, \ldots, N\}$ that belong to one of the following two sets:

- For the subsystems Σ_i, $i \in \mathcal{O} \subset \mathcal{N}$, the state $\boldsymbol{x}_i$ is measurable.
- For the subsystems Σ_i, $i \in \mathcal{D} = \mathcal{N} \setminus \mathcal{O}$, the state $\boldsymbol{x}_i$ is not accessible for measurement and the system matrix $\boldsymbol{A}_i$ of Σ_i is Hurwitz.

Hence, $\mathcal{O}$ and $\mathcal{D}$ are disjoint and $\mathcal{O} \cup \mathcal{D} = \mathcal{N}$. The question to be answered in this paper concerns the immeasurable states: Can the deviation of the immeasurable states from the corresponding states of the reference system be bounded by an event-based controller that has access only to the measurable subsystem states such that a desired performance is guaranteed for the overall system?

Figure 1 illustrates the structure of the event-based control loop, which consists of the plant with N physically interconnected subsystems, the event generators (EG), the control input generators (CIG) and the communication network. The solid lines represent a continuous information transmission, whereas the dashed lines indicate a communication at the event times t_k $(k = 0, 1, \ldots)$ where k denotes the event counter. In the investigated event-based control approach the triggering condition in the event generator of Σ_i depends upon the plant state $\boldsymbol{x}_i$. Since this information is not accessible for Σ_i $(i \in \mathcal{D})$, no event generator can be applied to the respective subsystem. This case is illustrated in Fig. 1 for Σ_N where N is considered to be an element of $\mathcal{D}$. The figure shows that the constraints on the measurability of the subsystem states implies some restrictions on the control structure.

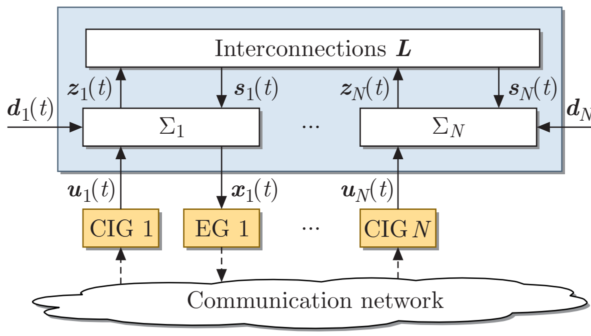

Fig. 1. Structure of the event-based control loop

The paper will show that, despite the constraints on the state measurement, a desired control performance with respect to the disturbance rejection behavior can be achieved by appropriately adjusting the triggering conditions in the event generators i $(i \in \mathcal{O})$. In other words, the loss of information about the states $\boldsymbol{x}_j$ for all $j \in \mathcal{D}$ can be compensated by refining the accessible information $\boldsymbol{x}_i$ from Σ_i for all $i \in \mathcal{O}$. The method for the design of the triggering conditions will be formulated as a linear programming problem.

1.2 Literature review

Most event-based control approaches known from literature rely on full state-feedback information, like in the work of Grüne and Müller (2009), Heemels et al. (2007),

⋆ This work was supported by the German Research Foundation (DFG) within the Priority Program 1305 "Control Theory of Digitally Networked Dynamical Systems".

Lunze and Lehmann (2010), Stöcker and Lunze (2011), Tabuada (2007) who dealt with centralized event-based control strategies or Mazo and Tabuada (2011), De Persis et al. (2011), Wang and Lemmon (2008) who investigated decentralized and distributed control approaches or, moreover, Anta and Tabuada (2010) who proposed a method for the self-triggered implementation of a class of event-based state-feedback control approaches. However, the availability of full state information is a strong assumption which is often not fulfilled in practice. Therefore, recently some effort has been spent on the investigation of event-based output-feedback control concepts. An output-based event-triggered controller has been proposed by Donkers and Heemels (2010). Trimpe and D'Andrea (2011) studied a continuous state-feedback controller that receives information from an event-based state estimator. Further approaches to event-based state estimators have been presented by Xu and Hespanha (2004) and Weimer et al. (2012). In contrast to this, several approaches use the output in order to determine an estimation of the current plant state that, instead of the actual state, is then applied to an event-based state-feedback controller. This strategy has been pursued by Lichun and Lemmon (2010) and Lehmann and Lunze (2011) where a Kalman filter or a Luenberger observer, respectively, has been used to extend the existing event-based controller.

Both previously mentioned approaches consider the plant to be completely observable. The assumptions made in this paper are less stringent, as they allow the subsystems Σ_i $(i \in \mathcal{D})$ to be unobservable. Note that this paper considers the states of the subsystems Σ_i $(i \in \mathcal{O})$ to be measurable. On condition that these subsystems are observable, this assumption can be relaxed by using state estimators for the subsystems Σ_i $(i \in \mathcal{O})$, following the ideas of Lehmann and Lunze (2011) or Lichun and Lemmon (2010).

1.3 Notation

$s \in \mathbb{R}$ denotes a scalar and $s \in \mathbb{R}_+$ means $s > 0$. $\boldsymbol{v} \in \mathbb{R}^{\nu}$ denotes a vector with the elements $v_1 \dots v_\nu$. $\boldsymbol{M} \in \mathbb{R}^{\mu\times\sigma}$ is a matrix where

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}_{1\bullet} \\ \vdots \\ \boldsymbol{M}_{N\bullet} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{11} & \dots & \boldsymbol{M}_{1N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{N1} & \dots & \boldsymbol{M}_{NN} \end{pmatrix}.$$

$\boldsymbol{M}_{i\bullet} \in \mathbb{R}^{\mu_i\times\sigma}$ and $\boldsymbol{M}_{ij} \in \mathbb{R}^{\mu_i\times\sigma_j}$ are submatrices of appropriate dimension. A (block-)diagonal matrix $\boldsymbol{D}$ is abridged by $\boldsymbol{D} = \mathrm{diag}\,(\boldsymbol{D}_i)$ where $\boldsymbol{D}_i$ are the respective diagonal entries. $\boldsymbol{v}^\top$ indicates the transpose of the vector $\boldsymbol{v}$. A comparison between two vectors $\boldsymbol{v}, \tilde{\boldsymbol{v}} \in \mathbb{R}^\nu$ is to be understood to hold element-wise, like $\boldsymbol{v} \leq \tilde{\boldsymbol{v}}$ amounts to $v_i \leq \tilde{v}_i$ for all $i = 1, \dots, \nu$. $\|\cdot\|$ denotes an arbitrary vector and induced matrix norm and, in particular, $\|\cdot\|_\infty$ represents the supremum norm.

1.4 Outline of the paper

Section 2 introduces the state-feedback approach to event-based control of interconnected systems and defines a measurement for the control performance with respect to the disturbance rejection behavior. Section 3 develops the method for the design of the triggering conditions of the event generators i $(i \in \mathcal{O})$, which ensures the compliance with a required control performance. In Sec. 4 the application of the design method is demonstrated and the behavior of the event-based control loop is illustrated using the example of an interconnected two-tank system.

2. EVENT-BASED CONTROL OF INTERCONNECTED SYSTEMS

2.1 Plant

The overall plant is decomposed into N subsystems Σ_i, $i = 1, \dots, N$. Σ_i $(i \in \mathcal{N})$ is described by the linear state-space model

$$\Sigma_i : \begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}_i\boldsymbol{x}_i(t) + \boldsymbol{B}_i\boldsymbol{u}_i(t) + \boldsymbol{E}_i\boldsymbol{d}_i(t) + \boldsymbol{E}_{\mathrm{s}i}\boldsymbol{s}_i(t), \\ \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0}, \\ \boldsymbol{z}_i(t) = \boldsymbol{C}_{\mathrm{z}i}\boldsymbol{x}_i(t) \end{cases} \tag{1}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ denotes the state, $\boldsymbol{u}_i \in \mathbb{R}^{m_i}$ the control input and $\boldsymbol{d}_i \in \mathbb{R}^{p_i}$ the disturbance. $\boldsymbol{s}_i \in \mathbb{R}^{q_i}$ and $\boldsymbol{z}_i \in \mathbb{R}^{r_i}$ denote the coupling input or coupling output, respectively, and the subsystems are interconnected according to the relation

$$\boldsymbol{s}(t) = \boldsymbol{L}\boldsymbol{z}(t) \tag{2}$$

where the blocks $\boldsymbol{L}_{ii} = \boldsymbol{0}, (i = 1, \dots, N)$, i.e. the signal $\boldsymbol{s}_i(t)$ is not a function of $\boldsymbol{z}_i(t)$. The disturbance $\boldsymbol{d}_i(t)$ is assumed to be bounded by

$$\sup_{t\geq 0} \|\boldsymbol{d}_i(t)\| \leq d_{i\,\max}, \quad \forall i = 1, \dots, N. \tag{3}$$

2.2 Reference system

Following the idea of Lunze and Lehmann (2010), the main objective of the event-based controller is to mimic the behavior of a reference system, i.e. a continuous-time state-feedback control system. For the purpose of the reference system design assume that $\mathcal{O} = \mathcal{N}$, that is, the states of Σ_i for all $i = 1, \dots, N$ are measurable and there exists a decentralized state-feedback gain $\boldsymbol{K} = \mathrm{diag}\,(\boldsymbol{K}_i)$ that yields a desired disturbance rejection behavior for the overall system. Hence, the subsystems' dynamics of the reference system are described by

$$\Sigma_{\mathrm{r}i} : \begin{cases} \dot{\boldsymbol{x}}_{\mathrm{r}i}(t) = \bar{\boldsymbol{A}}_i\boldsymbol{x}_{\mathrm{r}i}(t) + \boldsymbol{E}_i\boldsymbol{d}_i(t) + \boldsymbol{E}_{\mathrm{s}i}\boldsymbol{s}_{\mathrm{r}i}(t), \\ \boldsymbol{x}_{\mathrm{r}i}(0) = \boldsymbol{x}_{i0} \\ \boldsymbol{z}_{\mathrm{r}i}(t) = \boldsymbol{C}_{\mathrm{z}i}\boldsymbol{x}_{\mathrm{r}i}(t) \end{cases} \tag{4}$$

with $\bar{\boldsymbol{A}}_i = (\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_i)$. The signals of the reference system are indicated with the index r in order to distinguish them from the signals that occur in the event-based control loop. The interconnection of the subsystems $\Sigma_{\mathrm{r}i}, i = 1, \dots, N$ according to (2) yields the overall reference system

$$\dot{\boldsymbol{x}}_{\mathrm{r}}(t) = \left(\bar{\boldsymbol{A}} + \boldsymbol{E}_{\mathrm{s}}\boldsymbol{L}\boldsymbol{C}_{\mathrm{z}}\right)\boldsymbol{x}_{\mathrm{r}}(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}_{\mathrm{r}}(0) = \boldsymbol{x}_0 \tag{5}$$

with state $\boldsymbol{x}_{\mathrm{r}} \in \mathbb{R}^n$, $n = \sum_{i=1}^N n_i$, disturbance $\boldsymbol{d} \in \mathbb{R}^p$, $p = \sum_{i=1}^N p_i$, initial condition $\boldsymbol{x}_0 = \left(\boldsymbol{x}_{01}^\top \dots \boldsymbol{x}_{0N}^\top\right)^\top$ and where

$$\bar{\boldsymbol{A}} = \mathrm{diag}\left(\bar{\boldsymbol{A}}_i\right), \quad \boldsymbol{E}_{\mathrm{s}} = \mathrm{diag}\left(\boldsymbol{E}_{\mathrm{s}i}\right),$$
$$\boldsymbol{C}_{\mathrm{z}} = \mathrm{diag}\left(\boldsymbol{C}_{\mathrm{z}i}\right), \quad \boldsymbol{E} = \mathrm{diag}\left(\boldsymbol{E}_i\right).$$

In line with the work Stöcker et al. (2012) the state-feedback $\boldsymbol{K}$ is assumed to stabilize the interconnected subsystems, hence, the matrix $(\bar{\boldsymbol{A}} + \boldsymbol{E}_{\mathrm{s}}\boldsymbol{L}\boldsymbol{C}_{\mathrm{z}})$ is Hurwitz.

The following explains the design of the control input generators and event generators of the event-based control loop that is required to approximate the disturbance rejection behavior of the reference system (5).

2.3 Control input generation

The control input generator i ($i \in \mathcal{N}$) uses the model

$$\Sigma_{\rm s}: \begin{cases} \dot{\boldsymbol{x}}_{\rm s}(t) &= \left(\bar{\boldsymbol{A}} + \boldsymbol{E}_{\rm s}\boldsymbol{L}\boldsymbol{C}_{\rm z}\right)\boldsymbol{x}_{\rm s}(t), \\ \boldsymbol{x}_{{\rm s}j}(t_k^+) &= \boldsymbol{x}_j(t_k), \qquad \text{(for some } j \in \mathcal{O}) \end{cases} \tag{6}$$

$$\boldsymbol{u}_i(t) = -\boldsymbol{K}_i\boldsymbol{x}_{{\rm s}i}(t). \tag{7}$$

of the reference system (5) with $\boldsymbol{d}(t) = \boldsymbol{0}$ to determine the control input $\boldsymbol{u}_i(t)$. In (6),

$$\boldsymbol{x}_{\rm s} = \left(\boldsymbol{x}_{\rm s1}^\top \ \ldots \ \boldsymbol{x}_{{\rm s}N}^\top\right)^\top \in \mathbb{R}^n$$

denotes the model state. Whenever some event generator j ($j \in \mathcal{O}$) triggers an event, the state $\boldsymbol{x}_j(t_k)$ is sent from the event generator to all control input generators that reset the j-th component $\boldsymbol{x}_{{\rm s}j}$ of the overall model state $\boldsymbol{x}_{\rm s}$ to the current plant state $\boldsymbol{x}_j(t_k)$. t_k denotes the time instant at which event generator j triggers an event and t_k^+ denotes the right limit of t_k. Note that the models (6) used in each control input generator are synchronized for all $t \geq 0$ due to a simultaneous state reset at every event time t_k.

2.4 Event generation

The event generator i ($i \in \mathcal{O}$) continuously determines the deviation

$$\boldsymbol{x}_{\Delta i}(t) = \boldsymbol{x}_i(t) - \boldsymbol{x}_{{\rm s}i}(t)$$

and triggers an event whenever the condition

$$\|\boldsymbol{x}_{\Delta i}(t_k)\| = \bar{e}_i \tag{8}$$

holds, where $\bar{e}_i \in \mathbb{R}_+$ denotes the event threshold. Due to the event generation and the reset of the model state $\boldsymbol{x}_{{\rm s}i}(t)$, the difference state $\boldsymbol{x}_{\Delta i}(t)$ is bounded from above by

$$\sup_{t\geq 0} \|\boldsymbol{x}_{\Delta i}(t)\| = \bar{e}_i, \quad \forall t \geq 0, \ i \in \mathcal{O}. \tag{9}$$

As will be shown, the setting of the event thresholds $\bar{e}_i$ for all $i \in \mathcal{O}$ affects the performance of the event-based control system with respect to the disturbance rejection behavior. Hence, these parameters cannot be chosen freely if a desired disturbance rejection is to be satisfied.

2.5 Performance of the event-based control system

The performance of the event-based control system (1), (2), (6)–(8) with respect to the disturbance rejection is measured by the deviation between the behavior of the event-based system and the reference system. The difference behavior is described by the state-space model

$$\dot{\boldsymbol{e}}(t) = \dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{x}}_{\rm r}(t) = \left(\bar{\boldsymbol{A}} + \boldsymbol{E}_{\rm s}\boldsymbol{L}\boldsymbol{C}_{\rm z}\right)\boldsymbol{e}(t) + \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}_\Delta(t),$$
$$\boldsymbol{e}(0) = \boldsymbol{0}$$

with $\boldsymbol{x}_\Delta(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t)$, which yields

$$\boldsymbol{e}(t) = \int_0^t {\rm e}^{\left(\bar{\boldsymbol{A}} + \boldsymbol{E}_{\rm s}\boldsymbol{L}\boldsymbol{C}_{\rm z}\right)(t-\tau)}\boldsymbol{B}\boldsymbol{K}\boldsymbol{x}_\Delta(\tau){\rm d}\tau.$$

From the last equation the inequality

$$\|\boldsymbol{e}(t)\| \leq \kappa \cdot \sup_{t\geq 0}\|\boldsymbol{x}_\Delta(t)\| \tag{10}$$

follows, where

$$\kappa = \int_0^\infty \left\| {\rm e}^{\left(\bar{\boldsymbol{A}} + \boldsymbol{E}_{\rm s}\boldsymbol{L}\boldsymbol{C}_{\rm z}\right)\tau}\boldsymbol{B}\boldsymbol{K}\right\| {\rm d}\tau.$$

Note that κ is finite since the matrix $(\bar{\boldsymbol{A}} + \boldsymbol{E}_{\rm s}\boldsymbol{L}\boldsymbol{C}_{\rm z})$ is Hurwitz. Inequality (10) shows that the deviation between the behavior of the event-based system and the reference system depends upon the deviation state $\boldsymbol{x}_\Delta(t)$. This result is summarized in the following lemma.

Lemma 1. The deviation $\boldsymbol{e}(t)$ between the event-based control system (1), (2), (6)–(8) and the reference system (5) is bounded from above according to (10).

Based on Lemma 1 the performance of the event-based control system with respect to its disturbance rejection behavior is defined as follows.

Definition 1. The maximum deviation between the event-based control system (1), (2), (6)–(8) and the reference system (5) is called performance of the event-based control system:

$$J = \sup_{t\geq 0}\|\boldsymbol{e}(t)\| \leq \kappa \cdot \sup_{t\geq 0}\|\boldsymbol{x}_\Delta(t)\|. \tag{11}$$

Note that a desired level of performance $\rho \in \mathbb{R}_+$ in terms of

$$J \leq \rho \quad \Leftrightarrow \quad \sup_{t\geq 0}\|\boldsymbol{x}_\Delta(t)\| \leq \frac{\rho}{\kappa} = \tilde{\rho} \tag{12}$$

can trivially be accomplished due to the event triggering (8) only if $\mathcal{O} = \mathcal{N}$, because then Eq. (9) holds for all $i = 1, \ldots, N$ and the event thresholds $\bar{e}_i$ can be chosen such that the condition (12) is satisfied. However, since $\mathcal{O} \subset \mathcal{N}$ is considered in this paper, the deviation states $\boldsymbol{x}_{\Delta i}(t)$ for all $i \in \mathcal{D}$ are not bounded by (9), which raises the question how a desired performance (12) can be achieved in this case. This question will be answered in the next section.

3. EVENT THRESHOLD DESIGN

3.1 Boundedness of the deviation states

The previous section showed that a requirement on the performance of the event-based control system (1), (2), (6)–(8) can be expressed as a condition on the boundedness of $\boldsymbol{x}_\Delta(t)$. The following analysis derives bounds on the deviation state $\boldsymbol{x}_{\Delta i}$ ($i \in \mathcal{D}$), which will be shown to be a function of the event thresholds $\bar{e}_l$ ($l \in \mathcal{O}$). This result will later be used to develop a design method for the event thresholds $\bar{e}_l$ that guarantee a desired control performance according to (12).

First consider that the behavior of the deviation state $\boldsymbol{x}_{\Delta i}(t)$ ($i \in \mathcal{D}$) is described by the state space model

$$\dot{\boldsymbol{x}}_{\Delta i}(t) = \dot{\boldsymbol{x}}_i(t) - \dot{\boldsymbol{x}}_{{\rm s}i}(t) = \boldsymbol{A}_i\boldsymbol{x}_{\Delta i}(t) + \boldsymbol{E}_i\boldsymbol{d}_i(t) + \boldsymbol{E}_{{\rm s}i}\boldsymbol{s}_{\Delta i}(t),$$
$$\boldsymbol{x}_{\Delta i}(0) = \boldsymbol{0} \tag{13}$$

where $\boldsymbol{s}_{\Delta i}(t) = \boldsymbol{s}_i(t) - \boldsymbol{s}_{{\rm s}i}(t)$ denotes the deviation between the coupling inputs $\boldsymbol{s}_i(t)$ of the plant and $\boldsymbol{s}_{{\rm s}i}(t)$ of the model:

$$\boldsymbol{s}_i(t) = \boldsymbol{L}_{i\bullet}\boldsymbol{C}_{\rm z}\boldsymbol{x}(t), \qquad \boldsymbol{s}_{{\rm s}i}(t) = \boldsymbol{L}_{i\bullet}\boldsymbol{C}_{\rm z}\boldsymbol{x}_{\rm s}(t). \tag{14}$$

With $\boldsymbol{H}_{i\bullet} = (\boldsymbol{E}_{{\rm s}i}\boldsymbol{L}_{i\bullet}\boldsymbol{C}_{\rm z})$ from (13), (14) then

$$\begin{aligned}\dot{\boldsymbol{x}}_{\Delta i}(t) &= \boldsymbol{A}_i\boldsymbol{x}_{\Delta i}(t) + \boldsymbol{E}_i\boldsymbol{d}_i(t) + \boldsymbol{H}_{i\bullet}\boldsymbol{x}_\Delta(t) \\ &= \boldsymbol{A}_i\boldsymbol{x}_{\Delta i}(t) + \boldsymbol{E}_i\boldsymbol{d}_i(t) + \sum_{j=1}^N \boldsymbol{H}_{ij}\boldsymbol{x}_{\Delta j}(t)\end{aligned} \tag{15}$$

follows. Note that $\boldsymbol{H}_{ii} = \mathbf{0}$, since $\boldsymbol{L}_{ii} = \mathbf{0}$. Equation (15) yields

$$\boldsymbol{x}_{\Delta i}(t) = \int_0^t \mathrm{e}^{\boldsymbol{A}_i(t-\tau)} \boldsymbol{E}_i \boldsymbol{d}_i(\tau)\mathrm{d}\tau + \sum_{j=1}^{N} \int_0^t \mathrm{e}^{\boldsymbol{A}_i(t-\tau)} \boldsymbol{H}_{ij} \boldsymbol{x}_{\Delta j}(\tau)\mathrm{d}\tau.$$

Let

$$\gamma_{\mathrm{d}i} = \int_0^\infty \left\| \mathrm{e}^{\boldsymbol{A}_i \tau} \boldsymbol{E}_i \right\| \mathrm{d}\tau, \quad \gamma_{ij} = \int_0^\infty \left\| \mathrm{e}^{\boldsymbol{A}_i \tau} \boldsymbol{H}_{ij} \right\| \mathrm{d}\tau,$$

then the deviation state $\boldsymbol{x}_{\Delta i}$ is bounded according to the following relation:

$$\sup_{t \geq 0} \|\boldsymbol{x}_{\Delta i}(t)\| \leq \gamma_{\mathrm{d}i} \cdot d_{i\,\max} + \sum_{j=1}^{N} \gamma_{ij} \cdot \sup_{t \geq 0} \|\boldsymbol{x}_{\Delta j}(t)\|.$$

To make this statement more precise, substitute (9) into the last inequality which yields

$$\sup_{t \geq 0} \|\boldsymbol{x}_{\Delta i}(t)\| \leq \gamma_{\mathrm{d}i} \cdot d_{i\,\max} + \sum_{l \in \mathcal{O}} \gamma_{il} \cdot \bar{e}_l + \sum_{j \in \mathcal{D} \setminus i} \gamma_{ij} \cdot \sup_{t \geq 0} \|\boldsymbol{x}_{\Delta j}(t)\|. \tag{16}$$

Note that (16) is an implicit expression for the bound on $\boldsymbol{x}_{\Delta i}$ as it is a function of the bounds on $\boldsymbol{x}_{\Delta j}$ for all $j \in \mathcal{D} \setminus \{i\}$. In order to arrive at an explicit formulation for the bounds on the deviation states let

$$\delta_i = \sup_{t \geq 0} \|\boldsymbol{x}_{\Delta i}(t)\|, \quad \text{for all } i \in \mathcal{D}. \tag{17}$$

In the following it is exploited that the overall plant can always be transformed such that $\mathcal{O} = \{1, \ldots, O\}$ and $\mathcal{D} = \{D, \ldots, N\}$ with $D = O + 1$ and where O is the number of subsystems where the state is measurable. Then (16) yields

$$\begin{pmatrix} \delta_D \\ \vdots \\ \delta_N \end{pmatrix} \leq \boldsymbol{\Gamma}_{\mathcal{D}}^{-1} \begin{pmatrix} \gamma_{\mathrm{d}D} \cdot d_{D\,\max} + \sum_{j \in \mathcal{O}} \gamma_{Dj} \cdot \bar{e}_j \\ \vdots \\ \gamma_{\mathrm{d}N} \cdot d_{N\,\max} + \sum_{j \in \mathcal{O}} \gamma_{Nj} \cdot \bar{e}_j \end{pmatrix}, \tag{18}$$

with

$$\boldsymbol{\Gamma}_{\mathcal{D}} = \begin{pmatrix} 1 & \ldots & -\gamma_{DN} \\ \vdots & \ddots & \vdots \\ -\gamma_{ND} & \ldots & 1 \end{pmatrix}. \tag{19}$$

This shows that the bounds δ_i can be manipulated by appropriately setting the event thresholds $\bar{e}_j$ $(j \in \mathcal{O})$. Note that $\boldsymbol{\Gamma}_{\mathcal{D}}^{-1}$ exists and is non-negative if $\boldsymbol{\Gamma}_{\mathcal{D}}$ is an M-Matrix (Lunze (1992)).

Lemma 2. The deviation states $\boldsymbol{x}_{\Delta i}$ for all $i \in \mathcal{D}$ are bounded from above by Eq. (18).

Remark 1. In Eq. (13) the initial condition $\boldsymbol{x}_{\Delta i}(0) = \mathbf{0}$ implies $\boldsymbol{x}_i(0) = \boldsymbol{x}_{\mathrm{s}i}(0)$. However, since the state $\boldsymbol{x}_i$ for $i \in \mathcal{D}$ cannot be measured, the exact initial condition $\boldsymbol{x}_i(0)$ of Σ_i is uncertain, which generally leads to $\boldsymbol{x}_{\Delta i}(0) \neq \mathbf{0}$. In this case the result (18) holds with

$$\delta_i = \limsup_{t \to \infty} \|\boldsymbol{x}_{\Delta i}(t)\|$$

instead of the definition made in (17).

3.2 Design method

The result summarized in Lemma 2 will now be used to develop a method for the design of the event thresholds $\bar{e}_j$ $(j \in \mathcal{O})$, such that a desired performance (12) is guaranteed. In the following the previously introduced norms are specified as supremum norm $\|\cdot\|_\infty$. Hence, the performance requirement (12) reduces to

$$\sup_{t \geq 0} \|\boldsymbol{x}_{\Delta i}(t)\|_\infty \leq \frac{\rho}{\kappa} = \tilde{\rho}, \quad \text{for all } i = 1, \ldots, N, \tag{20}$$

which together with (9), (17) implies that the design aim is to find event thresholds $\bar{e}_j$ that satisfy the relation

$$0 < \bar{e}_j \leq \tilde{\rho}, \qquad \text{for all } j \in \mathcal{O} \tag{21}$$

and which guarantee that

$$0 \leq \delta_i \leq \tilde{\rho} \qquad \text{for all } i \in \mathcal{D}$$

holds.

Let

$$\bar{\boldsymbol{e}} = (\bar{e}_1 \ \ldots \ \bar{e}_O)^\top, \ \text{ with } \mathcal{O} = \{1, \ldots, O\}$$
$$\boldsymbol{\delta} = (\delta_D \ \ldots \ \delta_N)^\top, \ \text{ with } \mathcal{D} = \{D, \ldots, N\}.$$

Then the following equation is equivalent to (18):

$$\boldsymbol{\Gamma}_{\mathcal{D}} \boldsymbol{\delta} \leq \boldsymbol{g} + \boldsymbol{\Gamma}_{\mathcal{O}} \ \bar{\boldsymbol{e}}$$

with $\boldsymbol{\Gamma}_{\mathcal{D}}$ according to (19) and

$$\boldsymbol{\Gamma}_{\mathcal{O}} = \begin{pmatrix} \gamma_{D1} & \ldots & \gamma_{DO} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \ldots & \gamma_{NO} \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} \gamma_{\mathrm{d}D} \cdot d_{D\,\max} \\ \vdots \\ \gamma_{\mathrm{d}N} \cdot d_{N\,\max} \end{pmatrix}.$$

Hence, the design aim is fulfilled if the event thresholds $\bar{\boldsymbol{e}}$ are chosen subject to (21) such that the following inequality

$$\boldsymbol{g} + \boldsymbol{\Gamma}_{\mathcal{O}} \ \bar{\boldsymbol{e}} \leq \boldsymbol{\Gamma}_{\mathcal{D}} \ \tilde{\boldsymbol{\rho}} \tag{22}$$

with $\tilde{\boldsymbol{\rho}} = (\tilde{\rho} \ \ldots \ \tilde{\rho})^\top$ is satisfied.

Theorem 1. The vector $\bar{\boldsymbol{e}}$ of event thresholds $\bar{e}_i$ $(i \in \mathcal{O})$ guarantees a desired level of performance (20) for the event-based control system (1), (2), (6)–(8) if it satisfies the inequalities (21), (22).

The conditions (21), (22) on the event thresholds $\bar{e}_i$ $(i \in \mathcal{O})$ will now be used to formulate the design method that can be stated as the linear programming problem

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \bar{\boldsymbol{e}} \\ \text{s.t.} \quad & \boldsymbol{\Gamma}_{\mathcal{O}} \ \bar{\boldsymbol{e}} \leq \boldsymbol{\Gamma}_{\mathcal{D}} \ \tilde{\boldsymbol{\rho}} - \boldsymbol{g}, \\ & \bar{\boldsymbol{e}} \leq \tilde{\boldsymbol{\rho}}, \\ & \bar{\boldsymbol{e}} > \mathbf{0} \end{aligned} \tag{23}$$

where $\boldsymbol{c}$ is a vector of weighting factors. Note that $\boldsymbol{c} > \mathbf{0}$ is a reasonable choice for the objective function, since this implies the maximization of the event threshold $\bar{e}_i$ $(i \in \mathcal{O})$. From (8) follows that an enlargement of the threshold $\bar{\boldsymbol{e}}_i$ leads to larger inter-event times $t_{k+1} - t_k$ with respect to event generation in Σ_i.

In due consideration of the constraint (21), the problem (23) is feasible only if (i) the subsystems Σ_i and Σ_j for all $i, j \in \mathcal{D}$ are weakly coupled and (ii) the disturbance $\boldsymbol{d}_i$ that affects Σ_i for $i \in \mathcal{D}$ is small with respect to the desired performance level $\tilde{\rho}$. These necessary conditions are now stated more precisely. Firstly, the right-hand side of (22) must be (element-wise) positive, since $\boldsymbol{g} \geq \mathbf{0}$ and

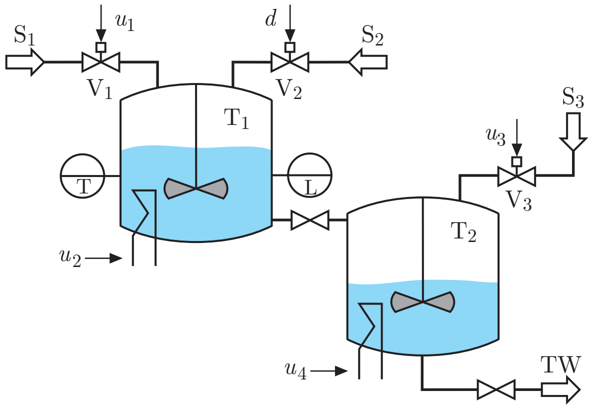

Fig. 2. Two-tank system

$\mathbf{\Gamma}_{\mathcal{O}} \geq \mathbf{0}$. This condition implies the demand for weak coupling between Σ_i and Σ_j for all $i, j \in \mathcal{D}$ in terms of

$$\sum_{j \in \mathcal{D}} \gamma_{ij} < 1, \quad \text{for all } i \in \mathcal{D}. \tag{24}$$

Secondly, a set of event thresholds $\bar{\boldsymbol{e}}$ that solves the problem (23) can be found only if

$$\boldsymbol{g} < \mathbf{\Gamma}_{\mathcal{D}}\, \tilde{\boldsymbol{\rho}} \tag{25}$$

holds. This condition is satisfied if the disturbance bounds $d_{i\,\max}$ $(i \in \mathcal{D})$ are sufficiently small.

4. EXAMPLE

4.1 Two-tank system

The proposed event-based control approach is now tested on a two-tank system which is depicted in Fig. 2. The process consists of two interconnected batch reactors T_1 and T_2, in each of which the level l and the temperature ϑ of the liquid shall be controlled. The level l_1 and temperature ϑ_1 of the content in Tank T_1 are measurable, whereas the level l_2 and temperature ϑ_2 in tank T_2 are neither measurable nor observable. The reactor T_1 is fed by the water supply S_1 via valve V_1 which can be controlled by means of the valve angle u_1. Analogously, tank T_2 is fed by the supply S_3. The valve angle u_3 is used to control the inflow. Heating rods in both reactors are applied to increase the temperature of the content. The process is disturbed by an additional and undesired inflow into tank T_1 from supply S_2, caused by a blockage of valve V_2. The maximum opening angle of valve V_2 is bounded by

$$|d(t)| \leq d_{\max} = 0.05. \tag{26}$$

For the following investigation, the overall system is decomposed into four scalar subsystems the states of which are $x_1 = l_1, x_2 = \vartheta_1, x_3 = l_2$ and $x_4 = \vartheta_2$. Each subsystem is described by a linear state-space model (1) with

$$\begin{aligned}
\boldsymbol{A}_1 &= -0.003, & \boldsymbol{B}_1 &= 0.14, & \boldsymbol{E}_1 &= 0.26,\\
\boldsymbol{A}_2 &= -0.006, & \boldsymbol{B}_2 &= 0.03, & \boldsymbol{E}_2 &= -0.11,\\
\boldsymbol{A}_3 &= -0.005, & \boldsymbol{B}_3 &= 0.13, & \boldsymbol{E}_3 &= 0,\\
\boldsymbol{A}_4 &= -0.006, & \boldsymbol{B}_4 &= 0.03, & \boldsymbol{E}_4 &= 0
\end{aligned}$$

and $\boldsymbol{E}_{si} = \boldsymbol{C}_{zi} = 1$ for all $i = 1, \ldots, 4$. The subsystems are interconnected according to the relation (2) with

$$\boldsymbol{L} = 10^{-3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3.76 & 0 & 0 & 0 \\ 5.62 & 0 & 0 & 0 \\ 0 & 4.71 & -4.16 & 0 \end{pmatrix}.$$

Note that the states of Σ_1 and Σ_2 are measurable, while the states of Σ_3 and Σ_4 are unknown, hence $\mathcal{O} = \{1, 2\}$ and $\mathcal{D} = \{3, 4\}$.

4.2 Controller and threshold design

For the continuous-time reference system the decentralized state-feedback gain $\boldsymbol{K} = \operatorname{diag}(0.1,\ 0.8,\ 0.2,\ 0.7)$ yields a desired disturbance rejection behavior. Therefore, $\boldsymbol{K}$ is also used in the event-based control system. The performance condition (12) is chosen to

$$J = \sup_{t \geq 0} \|\boldsymbol{e}(t)\|_\infty \leq \rho = 1.5. \tag{27}$$

According to (20) this requirement implies the constraint

$$\sup_{t \geq 0} \|x_{\Delta i}(t)\|_\infty \leq \tilde{\rho} = 1.33, \quad \forall i = 1, \ldots, 4. \tag{28}$$

In order to fulfill the stated performance requirements, the proposed event threshold design method is applied, which yields the event thresholds

$$\bar{e}_1 = 1.18, \qquad \bar{e}_2 = 0.55. \tag{29}$$

These results lead to the maximum deviations

$$\delta_3 = 1.33, \qquad \delta_4 = 1.33, \tag{30}$$

defined in (17), for the subsystems $3, 4 \in \mathcal{D}$.

4.3 Simulation results

The following analysis investigates the behavior of the event-based control loop subject to a constant disturbance $d(t) = 0.05$. The simulation results are illustrated in Fig. 3. The figures on the left-hand side show the level $x_1(t)$, the temperature $x_2(t)$ in tank T_1 and the events triggered by the event generator of Σ_1 (e_l) and of Σ_2 (e_ϑ). The figures on the right-hand side depict the level $x_3(t)$ and temperature $x_4(t)$ in tank T_2. In each of this figures the plant state $\boldsymbol{x}$ is represented by the solid line and the model state $\boldsymbol{x}_s$ by the dashed line. The disturbance $d(t)$ immediately affects subsystems Σ_1 and Σ_2 which reflects in the deviation between $x_1(t)$ and $x_{s1}(t)$ or $x_2(t)$ and $x_{s2}(t)$, respectively. An event is triggered in either of the the subsystems whenever the deviation state $x_{\Delta 1}(t)$ or $x_{\Delta 2}(t)$ reaches the respective event threshold specified in (29).

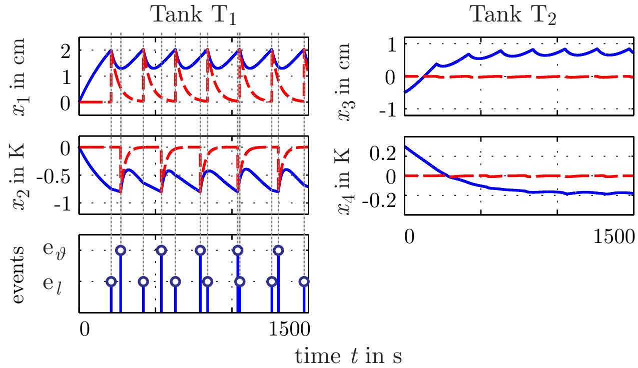

Fig. 3. Behavior of the event-based control system subject to a constant disturbance $d(t) = 0.05$

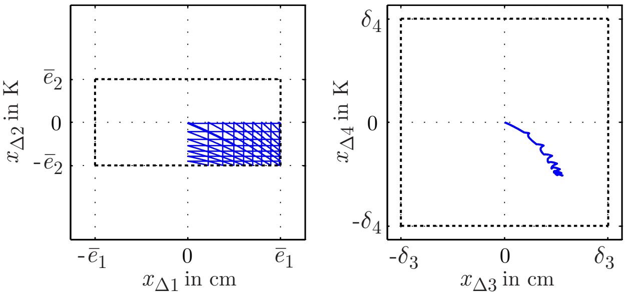

Fig. 4. Deviation state $\boldsymbol{x}_\Delta(t)$

The trajectory of the deviation state $\boldsymbol{x}_\Delta$ for each subsystem is illustrated in Fig. 4. The left-hand side of Fig. 4 shows the deviation state for Σ_1 and Σ_2. Each time one of the deviation states $x_{\Delta 1}(t)$ or $x_{\Delta 2}(t)$ reaches a bound of the rectangle denoting the event thresholds (29), the respective state is reset. For each subsystems the deviation state is obviously bounded due to the event generation and reinitialization. In contrast to this, the right-hand side of Fig. 4 shows the trajectory of the deviation states for the subsystems Σ_3 and Σ_4, where none of the deviation states reaches the rectangle that marks the bounds (30). This implies the compliance with the required performance according to (28) as can also be seen in Fig. 5.

Figure 5 shows the deviation $\|\boldsymbol{e}(t)\|$ between the event-based control system and the reference system. The solid line represents the case where the thresholds are set in accordance with (29) while the gray dotted line illustrates the deviation $\|\boldsymbol{e}(t)\|$ for the case where the thresholds are chosen freely to $\bar{e}_1 = \bar{e}_2 = 1.5$. It can be observed that the setting of the thresholds in the latter case leads to a violation of the performance requirement (27).

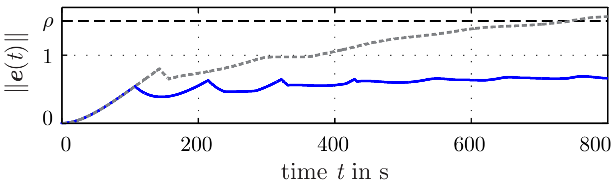

Fig. 5. Deviation between the event-based control system and the reference system. Solid line: $\bar{e}_1 = 1.18, \bar{e}_2 = 0.55$, dashed line: $\bar{e}_1 = 1.5, \bar{e}_2 = 1.5$

5. CONCLUSION

This paper proposed an event-based control approach for interconnected systems composed of subsystems, where the state is measurable only for some subsystems. A novel method for the design of the triggering conditions was derived and formulated as a linear programming problem, the solution of which guarantees a desired performance of the event-based control system with respect to the disturbance rejection behavior.

The proposed event-based control approach requires the model of the overall reference system to be included in each control input generator and event generator and is, therefore, storage and computing capacity consuming. While the application of this control approach to large-scale systems might be inefficient, this concept is suitable for systems where the state is not completely measurable and, moreover, the immeasurable part of the state cannot be reconstructed using an estimator, as done in the existing literature.

REFERENCES

Anta, A. and Tabuada, P. (2010). To sample or not to sample: Self-triggered control for nonlinear systems. *Automatica*, 55(9), 2030–2042.

De Persis, C., Sailer, R., and Wirth, F. (2011). On a small-gain approach to distributed event-triggered control. In *Proc. of 18th IFAC World Congress*, 2401–2406. Milan, Italy.

Donkers, M.C.F. and Heemels, W.P.M.H. (2010). Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved event-triggering. In *Proc. IEEE Conf. Decision & Control*, 3246–3251. Atlanta, GA, USA.

Grüne, L. and Müller, F. (2009). An algorithm for event-based optimal feedback control. In *Proc. IEEE Conf. Decision & Control*, 5311–5316. Shanghai, China.

Heemels, W.P.M.H., Sandee, J.H., and van den Bosch, P.P.J. (2007). Analysis of event-driven controllers for linear systems. *Int. J. Control*, 81(4), 571–590.

Lehmann, D. and Lunze, J. (2011). Event-based output-feedback control. In *Proc. Mediterranean Conf. Control & Automation*, 982–987. Corfu, Greece.

Lichun, L. and Lemmon, M.D. (2010). Event-triggered output feedback control of finite horizon discrete-time multi-dimension linear processes. In *Proc. IEEE Conf. Decision & Control*, 3221–3226. Atlanta, GA, USA.

Lunze, J. and Lehmann, D. (2010). A state-feedback approach to event-based control. *Automatica*, 46(1), 211–215.

Lunze, J. (1992). *Feedback Control of Large-Scale Systems*. Prentice Hall, London.

Mazo, M. and Tabuada, P. (2011). Decentralized event-triggered control over wireless sensor/actuator networks. *IEEE Trans. Autom. Control*, 56(10), 2456–2461.

Stöcker, C. and Lunze, J. (2011). Event-based control of nonlinear systems: An input-output linearization approach. In *Proc. IEEE Conf. Decision & Control*, 2541–2546. Orlando, FL, USA.

Stöcker, C., Lunze, J., and Vey, D. (2012). Stability analysis of interconnected event-based control loops. In *Proc. Conf. Analysis & Design of Hybrid Systems*, 58–63. Eindhoven, The Netherlands.

Tabuada, P. (2007). Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Trans. Autom. Control*, 52(9), 1680–1685.

Trimpe, S. and D'Andrea, R. (2011). Reduced communication state estimation for control of an unstable networked control system. In *Proc. IEEE Conf. Decision & Control*, 2361–2368. Orlando, FL, USA.

Wang, X. and Lemmon, M.D. (2008). Event design in event-triggered feedback control systems. In *Proc. IEEE Conf. Decision & Control*. Cancun, Mexico.

Weimer, J., Arajo, J., and Johansson, K.H. (2012). Distributed event-triggered estimation in networked systems. In *Proc. Conf. Analysis & Design of Hybrid Systems*, 178–185. Eindhoven, The Netherlands.

Xu, Y. and Hespanha, J. (2004). Optimal communication logics in networked control systems. In *Proc. IEEE Conf. Decision & Control*.